Motivic $C\tau$ Modules

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Homotopy theory for smooth schemes

Motivic homotopy theory is a homotopy theory for smooth schemes.

homotopy category of spaces	motivic homotopy category
smooth manifolds	smooth schemes
real line	affine line
topological spaces	simplicial sheaves over smooth schemes
homotopy invariance	$\mathbb{A}^1$-invariance
S^1	$S^{1,0}$, $\mathbb{G}_m$
stable homotopy category	stable motivic homotopy category

Motivic homotopy category

Let S be a Noetherian scheme, Sm/S be the category of smooth schemes of finite type over S.

Definition

The motivic homotopy category $Mot(S)$ is the homotopy localization of the ∞-topos of ∞-sheaves on the Nisnevich site of Sm/S with respect to the interval object $\mathbb{A}^1$.

Definition

The stable motivic homotopy category $SMot(S)$ is the stable ∞-category obtained from $Mot(S)$ by stabliztion with respect to $S^{1,1} = \mathbb{P}^1$.

Motivic Eilenberg-Mac Lane spaces

Let k be a field of characteristic 0. We can define the motivic Eilenberg-Mac Lane spaces over k as in the case for ordinary topological spaces:

The motivic Eilenberg-Mac Lane space $K(\mathbb{Z}(n), 2n) \in Mot(k)$ is the free abelien group object generated by the pointed motivic space $\mathbb{P}(k)^{\wedge n}$.

The motivic Eilenberg-Mac Lane spaces together form the motivic Eilenberg-Mac Lane spectrum $H^{mot}\mathbb{Z}$ which represents motivic homology.

Similarly we can define motivic Eilenberg-Mac Lane spectra with other coeficients such as $H^{mot}\mathbb{F}_p$.

Coeficient ring of motivic homology

The coeficient ring of motivic cohomology computed by Voevodsky in terms of Milnor K-theory.

In particular, over the field of complex numbers, we have:

Over the base field $\mathbb{C}$, the coeficient ring of mod p motivic homology is

$$H^{mot}\mathbb{F}_{p_{*,*}} = \mathbb{F}_p[\tau]$$

We will be working over $\mathbb{C}$ from now on.

The Betti realization functor

For a smooth scheme over $\mathbb{C}$, there is a canonical way to associate a complex manifold to it.

By the universal property of the motivic homotopy category, we get the Betti realization functor

$$re : Mot(\mathbb{C}) \to Top$$

which preserves homotopy colimits.

Moreover, the Betti realization functor stabilize to give the stable Betti realization functor

$$re : SMot(\mathbb{C}) \to Spectra$$

Comparison with classical homology

Denote by H^{mot} to be $H^{mot}\mathbb{F}_p$ and H for the classical mod p Eilenberg-Mac Lane spectrum $H\mathbb{F}_p$.

The Betti realization of H^{mot} is H. Moreover, there is a comparison map from the motivic homology to the classical homology of Betti realization.

For any motivic spectrum X, there is a natural map

$$H^{mot}_{*,*}(X)[\tau^{-1}] \to H_*(re(X)) \otimes \mathbb{F}_p[\tau^{\pm 1}]$$

which is an isomorphism if X is the sphere spectrum or more generally a cellular spectrum.

Motivic dual Steenrod algebra

The motivic dual Steenrod algebra, i.e. motivic homology of the motivic Eilenberg-Mac Lane spectrum, is computed by Voevodsky.

At the prime $p = 2$, we have:

$$H^{mot}_{*,*}H^{mot} = H^{mot}_{*,*}[\tau_0, \tau_1, \ldots, \xi_1, \xi_2, \ldots]/(\tau_k^2 = \tau\xi_{k+1})$$

The coaction is as follows:

$$\psi(\tau_k) = \tau_k \otimes 1 + \sum_{i=0}^{k} \xi_{k-i}^{2^i} \otimes \tau_i$$

$$\psi(\xi_k) = \sum_{i=0}^{k} \xi_{k-i}^{2^i} \otimes \xi_i$$

Motivic Adams spectral sequence

We can construct the Adams resolution using any motivic ring spetrum.

For motivic homology, we get the motivic Adams spectral sequence:

$$Ext_{H^{mot}_{*,*}H^{mot}}(H^{mot}_{*,*}, H^{mot}_{*,*}(X)) \Rightarrow \pi_{*,*}(X^{\wedge}_{H^{mot}})$$

converging conditionally for any motivic spectrum X.

Recall we are working over $\mathbb{C}$. In this case $\tau \in H^{mot}_{0,-1}$ is primitive.

Hence the motivic Adams spectral sequence shows the existence of an element

$$\tau \in \pi_{0,-1}(S^{mot})^{\wedge}_{H^{mot}}$$

Relation between classical and motivic ASS

We have a morphism of Hopf algebras

$$(H^{mot}_{*,*}, H^{mot}_{*,*}H^{mot}) \to (H_*, H_*H)$$

by sending τ to 1, and sending τ_i to ζ_{i+1}.

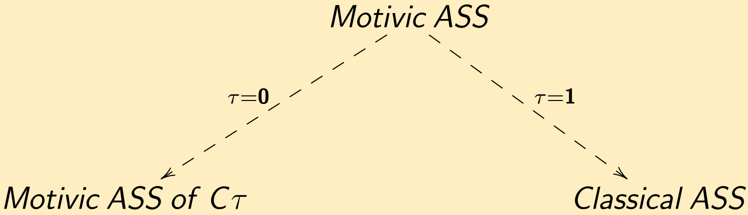
We call an object $X \in SMot$ cellular if it is a colimit of motivic spheres.

(Dugger-Isaksen)

Let $X \in SMot(\mathbb{C})$ be cellular. Then after inverting τ, the motivic Adams spectral sequence for X is isomorphic to the classical Adams spectral sequence for $re(X)$ tensored with $\mathbb{F}_p[\tau^{\pm}]$.

Motivic ASS as a deformation

Let $C\tau$ be the cofiber of the map $\tau : S^{0,-1} \to S^{0,0}$



$C\tau$ generates the fiber of Betti realization

We also have the following functors of stable infinity categories:

$$\begin{array}{ccccc} SMot_f^{\tau\text{-}tor} & \longrightarrow & SMot_f^{cell} & \longrightarrow & Spectra_f \\ \uparrow & & & & \\ C\tau\text{-}Mod_f & & & & \end{array}$$

The first row is an exact sequence of stable infinity categories.

Motivic ASS for $C\tau$ and ANSS

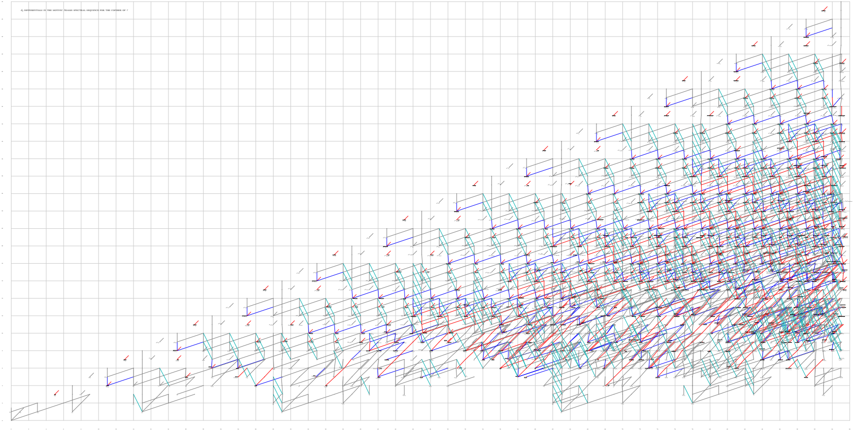
$$
\begin{array}{ccc}
Ext^{s,2w}_{BP_*BP}(BP_*, BP_*) & \xrightarrow{\cong} & \pi_{2w-s,w}(C\tau) \\
\Uparrow \text{ algebraic Novikov SS} & & \Uparrow \text{ motivic Adams SS} \\
Ext^{s,2w}_{P_*}(\mathbb{F}_p, I^{a-s}/I^{a-s+1}) & & Ext^{a,2w-s+a,w}_{\mathcal{A}^{mot}}(\mathbb{F}_p[\tau], \mathbb{F}_p)
\end{array}
$$

Wang's computer program

Isaksen's computation up to 59-stem

The same data!

E_2 term of motivic ASS for $C\tau$



$C\tau$-modules

The category of $C\tau$-modules can be studies with purely algebraic methods:

(Gheorghe-Wang-Xu)

There is an equivalence of stable infinity categories between the category of cellular $C\tau$-modules and the (unbounded) derived category of BP_*BP-comodules.

In particular, we can construct the Adams resolution of $C\tau$ using the algebraic Novikov filtration in BP_*BP-comodules, and get an isomorphism of spectral sequences:

(Gheorghe-Wang-Xu)

The algebraic Adams-Novikov spectral sequence is isomorphic to the motivic Adams spectral sequence for $C\tau$. Moreover, this isomorphism preserves all multiplicative structures, including Massey products.

Strategy of proof

1. Construct MU^{mot}/τ modules realizing injective MU_*MU-comodules using projective resolutions of MU_* modules.
2. Establish the general Adams-Novikov spectral sequence in $C\tau$-modules using injective resolutions of MU_*MU comodules.
3. Construct all mod τ Smith-Toda complexes.
4. Construct $C\tau$ modules realizing all BP_*BP comodules using the Landweber filtration theorem.
5. Establish the t-structure by induction on the Chow filtration.
6. Show the equivalence on the bounded derived category using Lurie's criterion.
7. Extend the equivalence using a filtered colimit argument.

Motivic $C\tau$ method

$$
\begin{array}{ccc}
Ext_{P_*}^{*,*}(\mathbb{F}_p, I^*/I^{*+1}) & \xRightarrow{\textbf{algebraic Novikov SS}} & Ext_{BP_*BP}^{*,*}(BP_*, BP_*) \\
\downarrow \cong & & \downarrow \cong \\
Ext_{H_{*,*}^{mot}H^{mot}}^{*,*,*}(\mathbb{F}_p[\tau], \mathbb{F}_p) & \xRightarrow{\textbf{motivic Adams SS}} & \pi_{*,*}(C\tau) \\
\uparrow & & \uparrow \\
Ext_{H_{*,*}^{mot}H^{mot}}^{*,*,*}(\mathbb{F}_p, \mathbb{F}_p) & \xRightarrow{\textbf{motivic Adams SS}} & \pi_{*,*}(S^{0,0}) \\
\downarrow & & \downarrow \\
Ext_{H_*H}^{*,*}(\mathbb{F}_p, \mathbb{F}_p) & \xRightarrow{\textbf{classical Adams SS}} & \pi_*(S^0)
\end{array}
$$

Motivic $C\tau$ method

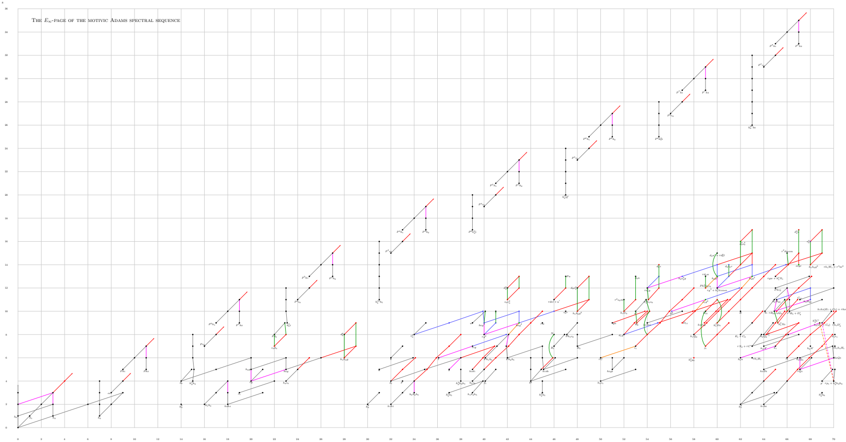
Algebraic Novikov d_r differentials (for any r)

$\longleftrightarrow$ Motivic Adams d_r differentials for $C\tau$

$\longrightarrow$ Motivic Adams $d_{r'}$ differentials for $S^{0,0}$ (for $r' \leq r$)

$\longrightarrow$ Classical Adams $d_{r'}$ differentials for S^0 (for $r' \leq r$)

E_∞-term of motivic Adams spectral sequence



Further questions

A natural question is: Does there exist an analogous theory over other fields?
Over a base field k. We can also construct a stable motivic category $SMot^{\text{ét}}(k)$ using the étale topology instead of the Nisnevich topology.
There is a natural functor

$$SMot(k) \to SMot^{\text{ét}}(k)$$

Question

Is there a subcategory C of $SMot(k)$, whose objects generate the kernel of the above functor, such that there exists a t-structure on C and C is equivalent to the derived category on its heart.